

Isomorphisms of Affine Plücker Spaces

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Abstract

All isomorphisms of Plücker spaces on affine spaces with dimensions ≥ 3 arise from collineations of the underlying affine spaces.

1 Introduction

Let L be a set and \sim a reflexive and symmetric binary relation on L such that (L, \sim) is connected, i.e., for any $a, b \in L$ there exists a finite sequence $a = a_1 \sim a_2 \sim \cdots \sim a_n = b$. Following W. BENZ the pair (L, \sim) is called a *Plücker space* [1, p. 199]. Elements $a, b \in L$ are said to be *related* if $a \sim b$. *Adjacent* elements ($a \approx b$) are characterized by $a \sim b$ and $a \neq b$.

The relation $\not\sim$ is reflexive and symmetric. However, $(L, \not\sim)$ is not necessarily a Plücker space, since it need not be connected. Nevertheless, L splits into a family of *connected components* with respect to $\not\sim$, say $(L_i)_{i \in I}$. Each component L_j ($j \in I$) gives rise to the Plücker space $(L_j, \not\sim_j)$, where $\not\sim_j$ denotes the restriction of $\not\sim$ to $L_j \times L_j$. On the other hand, L_j is not necessarily connected with respect to \sim_j , i.e., the restriction of \sim to $L_j \times L_j$. Hence (L_j, \sim_j) need not be a Plücker space.

Given two Plücker spaces (L, \sim) and (L', \sim') a bijection $\varphi : L \rightarrow L'$ is called an *isomorphism* if

$$a \sim b \iff a^\varphi \sim' b^\varphi \text{ for all } a, b \in L. \quad (1)$$

Obviously, (1) and

$$a \not\sim b \iff a^\varphi \not\sim' b^\varphi \text{ for all } a, b \in L \quad (2)$$

are equivalent conditions.

All automorphisms of (L, \sim) form its so-called *Plücker group*. Write, as above, $(L_i)_{i \in I}$ for the connected components of L with respect to $\not\sim$. Then each automorphism of $(L_j, \not\sim_j)$ ($j \in I$) extends to an automorphism of (L, \sim) by setting $x \mapsto x$ for all $x \in L \setminus L_j$.

Let $A = (\mathcal{P}, \mathcal{L}, \parallel)$ be an affine space, where \mathcal{P} , \mathcal{L} and \parallel denotes the set of points, the set of lines and the parallelism, respectively. Lines $a, b \in \mathcal{L}$ are called *related* ($a \sim b$), if $a \cap b \neq \emptyset$. The pair (\mathcal{L}, \sim) is satisfying the conditions mentioned before and will be called an *affine Plücker space*. We remark that for $\dim A \geq 3$ the set \mathcal{L} is the set of ‘points’ of partial linear space, the *affine Grassmann space* on \mathcal{L} ; cf. [4], [5], [6], [16] and [18]. However, the relation \sim is not the same as the binary relation of ‘collinearity’ used in those papers, since ‘collinear points’ are represented by lines that are related or parallel.

If $\dim A \neq 2$, then $(\mathcal{L}, \not\sim)$ is a Plücker space. If A is an affine plane, then $(\mathcal{L}, \not\sim)$ is not a Plücker space. The connected components $(\mathcal{L}_i)_{i \in I}$ with respect to $\not\sim$ are the pencils of parallel lines, since the relations $\not\sim$ and \parallel are coinciding now. If \mathcal{L}_j ($j \in I$) is a fixed pencil of parallel lines, then the relation $\not\sim_j$ is the coarsest relation on \mathcal{L}_j . Thus Plücker spaces on affine planes have indeed a very poor structure. The case $\dim A \leq 1$ cannot deserve interest at all.

We shall determine all isomorphisms of Plücker spaces on affine spaces A, A' with dimensions ≥ 3 : Any collineation yields an isomorphism of the associated affine Plücker spaces and vice versa. If we impose additional assumptions on A, A' (cf. Theorem 4), then any bijection $\varphi : \mathcal{L} \rightarrow \mathcal{L}'$ is already an isomorphism of Plücker spaces whenever (1) is satisfied with an implication (\implies) rather than an equivalence (\iff).

Similar theorems for Plücker spaces on projective spaces are due to W.L. CHOW [9], H. BRAUNER [7] and the author [12]. For further results and references on Plücker spaces see, among others, [1], [2] and [13].

2 Isomorphisms

Let $A = (\mathcal{P}, \mathcal{L}, \parallel)$ and $A' = (\mathcal{P}', \mathcal{L}', \parallel')$ be affine spaces. If $\kappa : \mathcal{P} \rightarrow \mathcal{P}'$ is a collineation, i.e., a bijection preserving collinearity and non-collinearity of points, then κ gives rise to a bijection

$$\varphi : \mathcal{L} \rightarrow \mathcal{L}', Q \vee R \mapsto Q^\kappa \vee R^\kappa \quad (Q, R \in \mathcal{P}, Q \neq R) \quad (3)$$

taking related lines to related lines in both directions.

We shall prove the following converse:

Theorem 1 *Let $A = (\mathcal{P}, \mathcal{L}, \parallel)$ and $A' = (\mathcal{P}', \mathcal{L}', \parallel')$ be affine spaces with $\dim A' \geq 3$. Suppose that $\varphi : \mathcal{L} \rightarrow \mathcal{L}'$ is an isomorphism of the Plücker space (\mathcal{L}, \sim) onto the Plücker space (\mathcal{L}', \sim') . Then*

$$\kappa : \mathcal{P} \rightarrow \mathcal{P}', a \cap b \mapsto a^\varphi \cap b^\varphi \quad (a, b \in \mathcal{L}, a \approx b) \quad (4)$$

is a well-defined collineation.

Proof. (a) We infer from $\dim A' \geq 3$ and the bijectivity of φ that $\#\mathcal{L} > 1$. Therefore $\dim A \geq 2$. With $Q \in \mathcal{P}$ write $\mathcal{L}(Q)$ for the star of lines with centre Q , i.e., the set of all lines in \mathcal{L} running through Q . Any star of lines is a maximal set of mutually related lines.

Suppose that $(\mathcal{L}(Q))^\varphi$ contains a trilateral spanning a plane $\mathcal{E}' \subset \mathcal{P}'$, say. All lines of $(\mathcal{L}(Q))^\varphi$ are mutually related. Therefore they are all contained in \mathcal{E}' . By $\dim A' \geq 3$, there exists a line $a \in \mathcal{L}$ with $a^\varphi \cap \mathcal{E}' = \emptyset$. Thus

$$a^\varphi \not\sim' x^\varphi \text{ for all } x \in \mathcal{L}(Q) \quad (5)$$

and therefore

$$a \not\sim x \text{ for all } x \in \mathcal{L}(Q). \quad (6)$$

On the other hand, there exists a line joining Q with an arbitrarily chosen point of the line a . This contradicts (6).

Thus we have established that $(\mathcal{L}(Q))^\varphi$ is a subset of a star of lines for any $Q \in \mathcal{P}$. It is obvious now that (4) is a well-defined mapping.

(b) Given a point $Q' \in \mathcal{P}'$ one may show as above that $(\mathcal{L}'(Q'))^{\varphi^{-1}}$ is a subset of a star of lines. Therefore κ is a surjection and under φ stars of lines go over to stars of lines in both directions.

If points $Q, R \in \mathcal{P}$ are distinct, then

$$\#(\mathcal{L}(Q) \cap \mathcal{L}(R)) = \#((\mathcal{L}(Q))^\varphi \cap (\mathcal{L}(R))^\varphi) = 1 \quad (7)$$

and $Q^\kappa \neq R^\kappa$, whence κ is injective.

Three mutually distinct points $Q, R, S \in \mathcal{P}$ are collinear if, and only if,

$$\#(\mathcal{L}(Q) \cap \mathcal{L}(R) \cap \mathcal{L}(S)) = 1. \quad (8)$$

This in turn is equivalent to the collinearity of $Q^\kappa, R^\kappa, S^\kappa \in \mathcal{P}'$. Hence κ is a collineation. \square

Remark 1 If the order of A' is greater than two or if $\dim A' \leq 2$, then any collineation $\mathcal{P} \rightarrow \mathcal{P}'$ is even an *affinity*, i.e. a collineation preserving parallelism in both directions. Otherwise, the existence of a collineation $\mathcal{P} \rightarrow \mathcal{P}'$ implies the existence of an affinity $\mathcal{P} \rightarrow \mathcal{P}'$; see [17, 32.5 and 40.4]. Hence for $\dim A' \geq 3$ we obtain all isomorphisms of (\mathcal{L}, \sim) onto (\mathcal{L}', \sim') via the Plücker group of (\mathcal{L}, \sim) and a single affinity $\mathcal{P} \rightarrow \mathcal{P}'$.

Remark 2 If $A = A'$, then Theorem 1 describes the Plücker group for affine spaces with dimension ≥ 3 . This generalizes a result in [1, p. 205] for real affine spaces¹.

¹The proof given there fails to work in case of characteristic two.

Remark 3 Suppose that $\dim A \geq 2$. The following construction yields all maximal sets of mutually related lines that are different from stars: Choose a point Q , an incident line a and a plane \mathcal{E} containing a . Write $\mathcal{L}(Q, \mathcal{E})$ for the pencil of lines in \mathcal{E} running through Q . Next define a family $(\tau_x)_{x \in \mathcal{L}(Q, \mathcal{E})}$ of translations $\tau_x : \mathcal{E} \rightarrow \mathcal{E}$ such that $\bigcap_{x \in \mathcal{L}(Q, \mathcal{E})} x^{\tau_x} = \emptyset$. Then $\{x^{\tau_x} \mid x \in \mathcal{L}(Q, \mathcal{E})\}$ is a maximal set of mutually related lines other than a star.

Remark 4 If $\varphi : \mathcal{L} \rightarrow \mathcal{L}'$ is an isomorphism and if $\dim A' = 2$, then $\dim A = 2$ according to Theorem 1. By virtue of (2) it is easy to establish the following result: Plücker spaces on affine planes A and A' are isomorphic if, and only if, A and A' have equipotent pencils of parallel lines or, in other words, if the order of A equals the order of A' .

The transposition of two distinct parallel lines of an affine plane is an example of a Plücker transformation that does not stem from a collineation.

We are now going to weaken the assumptions on φ in Theorem 1.

Theorem 2 Let $A = (\mathcal{P}, \mathcal{L}, \parallel)$ and $A' = (\mathcal{P}', \mathcal{L}', \parallel')$ be affine spaces with $\dim A' \geq 3$. Suppose that $\varphi : \mathcal{L} \rightarrow \mathcal{L}'$ is a bijection satisfying

$$a \sim b \implies a^\varphi \sim' b^\varphi \text{ for all } a, b \in \mathcal{L}. \quad (9)$$

Then

$$\lambda : \mathcal{P} \rightarrow \mathcal{P}', a \cap b \mapsto a^\varphi \cap b^\varphi \ (a, b \in \mathcal{L}, a \approx b) \quad (10)$$

is a well-defined injection that preserves collinearity and non-collinearity of points. Moreover,

$$\mathcal{L}(Q)^\varphi = \mathcal{L}'(Q^\lambda) \text{ for all } Q \in \mathcal{P}. \quad (11)$$

Proof. (a) By the proof of Theorem 1, part (a), the following assertions have been already verified: The dimension of A is ≥ 2 . For all $Q \in \mathcal{P}$ the set $(\mathcal{L}(Q))^\varphi$ is a subset of a star of lines, whence (10) is a well-defined mapping.

(b) Let $Q, R \in \mathcal{P}$ be distinct and assume that $Q^\lambda = R^\lambda$. Choose a line $c \in \mathcal{L} \setminus (\mathcal{L}(Q) \cup \mathcal{L}(R))$ and a point $S \in c$ such that Q, R, S are not collinear. Therefore $Q \vee S$, $R \vee S$ and c are three distinct concurrent lines. We deduce from (10) that

$$S^\lambda = (Q \vee S)^\varphi \cap (R \vee S)^\varphi = Q^\lambda = R^\lambda. \quad (12)$$

Consequently, $Q^\lambda \in x^\varphi$ for all $x \in \mathcal{L}$. This is impossible due to the surjectivity of φ . Hence λ is injective.

If $\{Q, R, S\} \subset \mathcal{P}$ is a triangle, then $Q \vee R$, $R \vee S$ and $S \vee Q$ are three distinct lines. The injectivity of φ and the injectivity of λ force that $\{Q^\lambda, R^\lambda, S^\lambda\} \subset \mathcal{P}'$ is a triangle. By definition, λ is a collinearity-preserving mapping.

(c) Finally, we establish (11). Assume to the contrary that there exists a point $Q \in \mathcal{P}$ and a line $b \in \mathcal{L} \setminus \mathcal{L}(Q)$ with $Q^\lambda \in b^\varphi$. Choose two distinct points $R_1, R_2 \in b$. Then $\{Q, R_1, R_2\}$ is a triangle, but $\{Q^\lambda, R_1^\lambda, R_2^\lambda\} \subset b^\varphi$ is a collinear set, an absurdity. \square

The aim of the following discussion is to give sufficient conditions for λ to be a collineation or, equivalently, a surjection. We could apply results on injective mappings of affine spaces preserving collinearity of points; see [1, 3.1–3.3], [19] and the references in [11]. However, we proceed instead in close analogy with [12]. Any star of lines in an affine space, for example, a star $\mathcal{L}(Q)$ in \mathbf{A} , carries in a natural way the structure of a projective space, viz.

$$\mathbf{A}/Q := (\mathcal{L}(Q), \{\mathcal{L}(Q, \mathcal{E}) \mid Q \in \mathcal{E}, \mathcal{E} \text{ a plane}\}). \quad (13)$$

Recall the following concept due to P.V. CECCHERINI [8]: A *semicollineation* of projective spaces is a bijection taking any three collinear points to collinear points. The existence of *proper* semicollineations (different from collineations) of Desarguesian projective spaces seems to be an open problem; see also [3], [14] and [15]. Semicollineations fall within the wider class of *weak linear mappings* that have been characterized independently in [10] and [11].

Now (11) can be improved as follows:

Theorem 3 *Let \mathbf{A} , \mathbf{A}' and φ be given as in Theorem 2. Then $\dim \mathbf{A} \geq 3$. If moreover the order of \mathbf{A} is not two and if $Q \in \mathcal{P}$, then the restricted mapping*

$$\varphi|_{\mathcal{L}(Q)} : \mathcal{L}(Q) \rightarrow \mathcal{L}'(Q^\lambda) \quad (14)$$

is a semicollineation of \mathbf{A}/Q onto \mathbf{A}'/Q^λ .

Proof. \mathbf{A} cannot be the affine plane of order two, since $6 < \#\mathcal{L}'$.

Suppose that the order of \mathbf{A} is not two. By (11), the mapping (14) is bijective. Let $a, b, c \in \mathcal{L}(Q)$ be ‘collinear points’ of \mathbf{A}/Q . There exists a line $d \in \mathcal{L} \setminus \mathcal{L}(Q)$ that is adjacent to a, b and c . Hence $a^\varphi, b^\varphi, c^\varphi \in \mathcal{L}(Q^\lambda, Q^\lambda \vee d^\varphi)$ represent ‘collinear points’ in \mathbf{A}'/Q^λ so that (14) is a semicollineation. There are non-coplanar lines through Q^λ representing ‘non-collinear’ points of \mathbf{A}'/Q^λ . Their pre-images under (14) are distinct and non-coplanar, so that $\dim \mathbf{A} \geq 3$. \square

If \mathbf{A} is of order two, then (14) is in general merely a bijection.

Theorem 4 *With the settings of Theorem 2, each of the following conditions is sufficient for λ to be a collineation:*

1. \mathbf{A} or \mathbf{A}' is a finite affine space.

2. $\dim A \leq \dim A' < \infty$.

3. The order of A is different from two and every monomorphism of an underlying field F of A in an underlying field F' of A' is surjective.

4. A and A' are affine spaces of order two.

Proof. *Ad 1.* Since φ is bijective, both A and A' are finite affine spaces.

Let $A \cong \text{AG}(n, 2)$ and $A' \cong \text{AG}(m, p^h)$, where $n \geq 3$, $m \geq 3$, $h \geq 1$ are integers and p is a prime. Choose $Q \in \mathcal{P}$. We deduce from (11) that

$$2^{n-1} \# \mathcal{L}(Q) = \# \mathcal{L} = \# \mathcal{L}' = p^{h(m-1)} \# \mathcal{L}(Q^\lambda) = p^{h(m-1)} \# \mathcal{L}(Q). \quad (15)$$

Consequently, $p = 2$, $n - 1 = h(m - 1)$ and, by $\# \mathcal{L}(Q) = \# \mathcal{L}(Q^\lambda)$,

$$\sum_{i=0}^{n-1} 2^i = \sum_{i=0}^{m-1} 2^{hi}. \quad (16)$$

We infer that each summand on the right hand side of (16) appears exactly once on the left hand side, whence $h = 1$ and $n = m$. This implies that λ is surjective.

If the order of A is greater than two, then (14) is a semicollineation and, by [8, 14.2], even a collineation. Therefore A and A' are of equal order. Now λ turns out to be surjective, because of

$$\dim A = \dim(A/Q) + 1 = \dim(A'/Q^\lambda) + 1 = \dim A' < \infty. \quad (17)$$

Ad 2. By virtue of the previous result, we may exclude affine spaces A , A' of order two from the following discussion.

Choose any point $R' \in \mathcal{P}'$. As φ is surjective, there exists a line $d \in \mathcal{L}$ with $R' \in d^\varphi$. Let $Q \in \mathcal{P}$ be off the line d and put $\mathcal{E} := Q \vee d$. We observe that

$$\dim(A/Q) = \dim A - 1 \leq \dim A' - 1 = \dim(A'/Q^\lambda) < \infty. \quad (18)$$

By [8, 8.4] or [14, Theorem 2.2] the semicollineation (14) turns out to be a collineation. Consequently, $\mathcal{L}(Q, \mathcal{E})^\varphi$ is a pencil of lines $\mathcal{L}'(Q^\lambda, \mathcal{E}')$, say. The line $d^\varphi \not\propto Q^\lambda$ is related to all lines of this pencil with at most one exception. This implies that the point-set d^λ is equal to the affine line d^φ . Thus

$$R' \in d^\varphi = d^\lambda \subset \text{im } \lambda. \quad (19)$$

Ad 3. Choose any point $Q \in \mathcal{P}$. By Theorem 3, the mapping (14) is a semicollineation. This implies the existence of a monomorphism $F \rightarrow F'$; cf. [8, 5.1], [10, Theorem 5.4.1] or [11, Theorem 2]. By [8, 5.3], the mapping (14) is a collineation. From this the surjectivity of λ is established as above.

Ad 4. Choose any point $R' \in \mathcal{P}'$. Since φ is surjective, there exists a line $\{R_1, R_2\} \in \mathcal{L}$ with $R' \in \{R_1, R_2\}^\varphi = \{R_1^\lambda, R_2^\lambda\}$. \square

Remark 5 Let A be an affine space over $\text{GF}(2)$ with a countable basis and let A' be an m -dimensional affine space ($3 \leq m \leq \aleph_0$) over a countable field F' of arbitrary characteristic. Hence there is either no monomorphism or no surjective monomorphism of $\text{GF}(2)$ in F' . As $\#\mathcal{P} = \#\mathcal{L}' = \aleph_0$, we can index all points of \mathcal{P} as Q_1, Q_2, \dots and all lines of \mathcal{L}' as a'_1, a'_2, \dots such that there are no repeated elements.

Let us define, by recursion, an injective sequence $\{1, 2, \dots\} \rightarrow \mathcal{P}'$, $s \mapsto R'_s$ such that each line of \mathcal{L}' contains exactly two points: We start with a point $R'_1 \in a'_1$ and put $\mathcal{B}'_1 := \{R'_1\}$. Next assume that we are already given a set $\mathcal{B}'_i = \{R'_1, \dots, R'_i\}$ formed by $i \geq 1$ mutually distinct points no three of which are collinear. Write \mathcal{N}'_i for the set of all lines that arise by joining distinct points of \mathcal{B}'_i . Then let $j \in \{1, 2, \dots\}$ be the least element such that the line a'_j is not in \mathcal{N}'_i . Since a'_j carries an infinite number of points, we can choose such a point $R'_{i+1} \in a'_j \setminus \mathcal{B}'_i$ that no three elements of the set $\mathcal{B}'_{i+1} := \mathcal{B}'_i \cup \{R'_{i+1}\}$ are collinear.

Put $\mathcal{B}' := \bigcup_{s=1}^{\infty} \mathcal{B}'_s$. By construction, no three distinct points of \mathcal{B}' are collinear. Furthermore, given a line $a'_k \in \mathcal{L}'$ we obtain that $a'_k \in \mathcal{N}'_{2k}$, as required.

The mapping

$$\varphi : \mathcal{L} \rightarrow \mathcal{L}', \{Q_s, Q_t\} \mapsto R'_s \vee R'_t, \quad (s, t \in \{1, 2, \dots\}, s \neq t) \quad (20)$$

is a bijection satisfying (9). The associated injection λ (see (10)) takes Q_s to R'_s ($s \in \{1, 2, \dots\}$). Only two points of a'_1 belong to $\text{im } \lambda$, whence λ is not surjective.

Remark 6 Let A, A' be affine spaces with equal infinite order and $2 = \dim A' < \dim A \leq \aleph_0$. There exists a bijection $\varphi : \mathcal{L} \rightarrow \mathcal{L}'$ such that any class of parallel lines in A is mapped onto a pencil of parallel lines in A' . Such a φ is satisfying (9) without being an isomorphism of Plücker spaces.

Remark 7 Let $A = A'$ be an affine plane of infinite order. Choose two non-parallel lines $a, b \in \mathcal{L}$. There exists a bijection $\varphi : \mathcal{L} \rightarrow \mathcal{L}$ such that the parallel class of a is mapped onto the union of the parallel classes of a and b , whereas any other pencil of parallel lines is mapped onto a pencil of parallel lines. Then φ is satisfying (9) without being a Plücker transformation.

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